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ON RECURRENT RANDOM WALKS ON SEMIGROUPS

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ABSTRACT. Let μ be a regular Borel probability measure on a locally compact semigroup S and consider the right (resp. left) random walk on $D = \overline{\bigcup F^n}$, $F = \operatorname{Supp} \mu$, with transition function $P^n(x, B) = \mu^n(x^{-1}B)$ (resp. $\mu^n(Bx^{-1})$). These Markov chains can be represented as $Z_n = X_1 X_2 \cdots X_n$ (resp. $S_n = X_n X_{n-1} \cdots X_1$), X_i's independent μ-distributed with values in S defined on an infinite-sequence space $(\Pi_1^\infty S_i, P)$, $S_i = S$ for all i. Let R_r (resp. R_l) $= \{x \in D; P_x(Z_n(S_n) \in N_x \text{ i.o.}) = 1$ for all neighborhoods N_x of $x\}$ and R'_r $(R'_l) = \{x \in D; P(Z_n(S_n) \in N_x \text{ i.o.}) = 1$ for all N_x of $x\}$. Let S be completely simple ($= E \times G \times F$, usual Rees product) in the results (1), (2), (3), (4), (5) below: (1) $x \in R_r$ *iff* $\sum \mu^n(x^{-1}N_x) = \infty$ for all neighborhoods N_x of x *iff* $\sum \mu^n(N_x) = \infty$ *for all* N_x *of* x. (2) Either $R_r = R_l = \varnothing$ or $R_r = R_l = D =$ also completely simple. (3) If the group factor G is compact, then there are recurrent values and we have $R_r = R_l = D =$ completely simple. (4) $R'_r \neq \varnothing$ *implies* $R'_r = R_r = R_l = D =$ a right subgroup of S (but R'_l may be $\varnothing$). (5) S can support a recurrent random walk (i.e., a r. walk with $R_r \neq \varnothing$) *iff* G ($=$ the group factor) can support a recurrent random walk. Finally (6) if S is compact abelian, then always $R' = R = K =$ the kernel of S. These results extend previously known results of Chung and Fuchs and Loynes.

1. Introduction. In this paper we introduce the notion of a recurrent random walk on a topological semigroup and establish results parallel to those given in [3] and [6]. In §2 we find that the definition of recurrence used in [3] and [6] is not suitable for semigroups. In §3, using a modified definition, we deal with random walks on locally compact completely simple semigroups which have been shown to be important in the study of random walks on semigroups [7], [8], [9], [10]. The problem of recurrence on general compact semigroups is considered in [11]. A few remarks are mentioned in §4 where general compact semigroups are also considered. Some definitions and notations used later on are given in the following paragraphs.

Let S be a locally compact Hausdorff topological semigroup and μ be a regular Borel probability measure with support $\Phi \subset S$. Let $X_1, X_2, \ldots$ be a sequence of independent random variables on some probability space (Ω, Σ, P) with values in S, having the same distribution.

$$P(X \in B) = \mu(B), \quad \text{for any Borel } B \subset S.$$

(As usual, we identify the process X_n with the usual coordinate representation

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process in the sequence space $S^\infty \equiv \Omega$, having the same distribution.) The sequence $Z_n = X_1X_2\cdots X_n$, $n = 1, 2, \ldots$, is called a *right random walk* on S generated by μ. (A *left random walk* is similarly defined.) The set

$$D = \overline{\bigcup_{n=1}^{\infty}\Phi^n}, \quad \text{the closed subsemigroup generated by } \Phi,$$

is called the *support of the random walk* $\{Z_n\}$. By the *kernel K* of D we shall mean the minimal (if it exists) two-sided ideal of D. We shall use the following notation:

$A^{-1}B \equiv \{s \in S; as \in B \text{ for some } a \in A\}$,
$x^{-1}B \equiv \{s \in S; xs \in B\}$. Also,
$Z_k^{-1}Z_n \equiv X_{k+1}X_{k+2}\cdots X_n$.

We have for Borel B, $P(Z_n \in B) = \mu^n(B)$, where $\mu^n = \mu * \mu * \cdots * \mu$ (n times). It can be easily verified that $P(Z_{n+k} \in B \mid Z_n = x) = \mu^k(x^{-1}B) =$ the kth transition function for Z_n. [We assume Z_n's to be measurable.]

The semigroup S is said to be *completely simple* if S is homeomorphic and isomorphic to a topological product $S \equiv E \times G \times F$ where E, F are left-zero and right-zero semigroups resp., G is a locally compact group and the multiplication of elements in $E \times G \times F$ is defined by

$$(e,g,f)\cdot(e',g',f') = (e,g(fe')g',f').$$

If $\bar{e} \in \mathcal{E}(S) \equiv$ the set of idempotent elements of S, then in the above representation we may (and do) choose $E \equiv \mathcal{E}(S\bar{e})$, $G \equiv \bar{e}S\bar{e}$, $F \equiv \mathcal{E}(\bar{e}S)$, so that $FE \subset G$ and the homeomorphism is given by

$$\eta\colon E\times G\times F \to S, \qquad \eta(e,g,f) = egf,$$

$$\eta^{-1}\colon S \to E\times G\times F, \qquad \eta^{-1}(s) = (s(\bar{e}s\bar{e})^{-1}, \bar{e}s\bar{e}, (\bar{e}s\bar{e})^{-1}s)$$

(see [1, pp. 46, 61] and [10]). If S is isomorphic to $G \times F$ where the multiplication of elements in $G \times F$ is reduced to direct-product multiplication $(g,f)\cdot(g',f') = (gg',f')$, then S is called a *right group.* (A *left group* is similarly defined.) A right group is right simple ($xS = S$ for all $x \in S$) and left cancellative and every idempotent is a left identity in S. We observe that if S is completely simple and $x = (e,g,f) \in S$, then $xS = \{e\} \times G \times F$ is a right group [1, p. 46].

2. **Preliminaries.** Suppose $X_1, X_2, \ldots$ are independent identically distributed random variables with values in a locally compact group G. In [3] and [6] an element $g \in G$ is defined to be *recurrent* for the right random walk $Z_n = X_1X_2\cdots X_n$ if

(1) $P(Z_n \in N_g$ infinitely often (i.o.)) $\equiv P(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{Z_k \in N_g\}) = 1$, for all open neighborhoods N_g of g. Let $R' \equiv$ the set of recurrent elements in the sense of (1).

An element $g \in G$ is called *possible* if for each N_g there exists a k such that $P(Z_k \in N_g) > 0$. It was proved that

(i) either there is no recurrent element or the set of all possible values are recurrent and form a closed subgroup of G;

(ii) there are recurrent values if and only if $\sum P(Z_n \in N) = \infty$ for all neighborhoods N of the identity of G.

The following example shows that condition (1) is not natural for random walks on semigroups.

Example 2.1. Let $S = E \times G$ be a left group where the group $G = \{1, a\}$ and the left-zero semigroup $E = \{e_1, e_2\}$ so that $e_1 e_2 = e_1^2 = e_1$, $e_2 e_1 = e_2^2 = e_2$. Let $\mu\{(e,g)\} = \frac{1}{4}$ for each $(e,g) \in S$. Consider the right random walk generated by μ, $Z_n = X_1 X_2 \cdots X_n$. Since $P(Z_n = (e_1, 1) \text{ i.o.} \mid Z_1 = (e_1, 1)) = 1$, we should say that $(e_1, 1)$ is recurrent. However, according to (1), $(e_1, 1)$ is not recurrent because $P(Z_n = (e_1, 1) \text{ i.o.}) = \frac{1}{2} < 1$.

This is due to the fact that a random walk on semigroups may have disjoint nonempty communicating classes;(3) certainly this cannot happen in groups. Also we cannot expect statement (i) above, to be true in semigroups because there exist nonessential(3) classes; this can be seen in the following example (which shows that not every possible state is recurrent).

Example 2.2. Let $S = [0, 1]$ with the Euclidean topology and with the multiplication of real numbers as the binary operation. Let μ be the Lebesgue measure on S. Then, $\mu((k, 1]x^{-1}) \leq \mu((k, 1])$, for $k > 0$; $\mu^2((k, 1]) = \int_k^1 \mu((k, 1]x^{-1})\mu(dx) \leq (1 - k)^2$, and by induction $\mu^n((k, 1]) \leq (1 - k)^n$, for all $k > 0$. Hence $\sum \mu^n((k,1]) < \infty$ which implies that $P(Z_n \in (k, 1] \text{ i.o.}) = 0$. Hence $P(Z_n \in [0, k) \text{ i.o.}) = 1$ for all $k > 0$. Therefore 0 is the only recurrent state for Z_n, while every $x \in (0, 1]$ is possible. It is easy to see that $(0, 1)$ and $\{1\}$ are nonessential communicating classes.

In the remaining part of this section, let R' be the set of recurrent states of $\{Z_n\}$ as defined by (1). Recall that D is the set of possible states for Z_n. We shall show below that if S is a completely simple semigroup then either $R' = \varnothing$ or $R' = D =$ a right subgroup of S, thus generalizing a corresponding result in [3] and [6].

Lemma 2.1. *Let S be a completely simple semigroup and let $R' \neq \varnothing$. Then D is contained in a sub-right-group of S.*

Proof. Suppose $(e,g,f) \in R'$ and $(e',g',f') \in D$ with $e \neq e'$. Then there exist neighborhoods N_e, $N_{e'}$ of e and e' in E respectively such that $N_e \cap N_{e'} = \varnothing$. Let N_g, $N_{g'}$ be neighborhoods of g and g' in G and N_f and $N_{f'}$ be neighborhoods in F. Since $(e',g',f') \in D$, we have $\mu^k(N_{e'} \times N_{g'} \times N_{f'}) > 0$ for some $k \geq 1$. Also

$$N_e \cap N_{e'} = \varnothing \Rightarrow (N_{e'} \times N_{g'} \times N_{f'})^{-1}(N_e \times N_g \times N_f) = \varnothing.$$

(3) These terms are used here in the sense of K. L. Chung, *Markov chains with stationary transition probabilities*, Springer-Verlag, Berlin, 1960, p. 11.

It follows that

$$P(Z_n \in N_e \times N_g \times N_f \text{ f.o. (finitely often)}) \geq \mu^k(N_{e'} \times N_{g'} \times N_{f'}) > 0.$$

Therefore $(e,g,f) \notin R'$, a contradiction. Hence there does not exist in D a state (e',g',f') with $e \neq e'$, i.e., $D \subset \{e\} \times G \times F$, a right group. Q.E.D.

Theorem 2.1. *If S is completely simple, then either $R' = \varnothing$ or $R' = D =$ a topological right group in S.*

Proof. Since D is inside a right group (Lemma 2.1) and since a right group with multiplication from the right is a group, the proof given in [3] and [6] can be applied in our case. In fact one may assume that D itself is a right group by Lemma 2.1; if $R' \neq \varnothing$, then every $d \in D$ has the property that $\sum \mu^n(N_d) = \infty$ for all N_d since the set of points with this property is an ideal and D is simple; then using similar arguments as in [6], it follows that if $(g, e_0) \in R'$ and $(c, e) \in D$, then $(c^{-1}g, e_0) \in R'$ and R' is a left ideal. Also by the argument in [6, Theorem 4, p. 452], every idempotent $e \in D$ is in R', from which one obtains $R' = D$. Q.E.D.

Following the arguments in [3] and [6] (see above proof) we can also show the following

Theorem 2.2. *If S is a right group, then $R' \neq \varnothing$ if and only if $\sum_{n=1}^{\infty} P(Z_n \in N_a) = \sum \mu^n(N_a) = \infty$ for all neighborhoods N_a of some $a \in D$.*

3. **The main results.** In this section we shall assume *that S is a locally compact, completely simple semigroup*, i.e., $S \equiv E \times G \times F$ as in §1. From the results in §2, we can see that it is more reasonable to define, for random walks on semigroups, a recurrent state as following:

Definition 3.1. We say that $x \in S$ is a *recurrent value* for the right random walk $Z_n = X_1 X_2 \cdots X_n$ on S if $x \in D$ and, for each neighborhood N_x of x,

$$P(Z_n \in N_x \text{ i.o.} \mid Z_1 = x) \equiv P_x(Z_n \in N_x \text{ i.o.}) = 1.$$

Notations. We write

(i) $P_x(E) \equiv P(E \mid X_1 = x)$.

(ii) $R \equiv \{x \in D; x \text{ is a recurrent value}\}$.

(iii) $x \to y$ if $x \in D$ and for each neighborhood N_y of y, $P_x(Z_n \in N_y) > 0$ for some $n \geq 1$. (Observe that $x \to y$ if and only if $y \in \overline{xD} = \text{cl}\,(xD)$.)

(iv) $x \to y$ i.o. if $x \in D$ and for each N_y, $P_x(Z_n \in N_y \text{ i.o.}) = 1$. (Observe that $x \to y$ i.o. $\Rightarrow y \in \overline{xD} \Rightarrow x \to y$.)

The following proposition of independent interest shows that for fixed elements $f, f' \in F$, whether $x \to y$ i.o. is independent of $e \in E$ and depends only on the product of the G-coordinates.

Proposition 3.1. *Let $x \to y$ i.o., in which case $y \in \overline{xD}$ and x, y must be of the form $x = (e, g_x, f)$, $y = (e, g_y, f')$ for some g_x, $g_y \in G$ and some $f, f' \in F$. Then* *for any* $z, w \in D$ *of the form* $z = (e', g_z, f)$, $w = (e', g_w, f')$, $z \to w$ *i.o. provided* $g_z^{-1} g_w = g_x^{-1} g_y$.

Proof. Let $N_y = N_e \times N_{g_y} \times N_{f'}$, where N_e, $N_{f'}$ are neighborhoods of e and f' in E and F respectively and $N_{g_y} \equiv g_y N_u$, N_u being a neighborhood of the unit element u in the group G. We note that

$$1 = P_x(Z_n \in N_y \text{ i.o.}) = P(xZ_n \in N_y \text{ i.o.}) = P(Z_n \in x^{-1} N_y \text{ i.o.})$$

and

$$\begin{aligned} x^{-1} N_y &= \{(a,b,c) \in S; (e, g_x, f) \cdot (a,b,c) \in N_y\} \\ &= \{(a,b,c) \in S; (e, g_x (fa) b, c) \in N_e \times g_y N_u \times N_{f'}\} \qquad (2) \\ &= \{(a,b,c) \in S; c \in N_{f'} \text{ and } b \in (fa)^{-1} g_x^{-1} g_y N_u\} \end{aligned}$$

which, given f, $N_{f'}$ and N_u, depends only on $g_x^{-1} g_y$. Hence, for $N_w = N_{e'} \times g_w N_u \times N_{f'}$ we have by a computation of $z^{-1} N_w$ as in (2), $x^{-1} N_y = z^{-1} N_w$ as long as $g_x^{-1} g_y = g_z^{-1} g_w$. Therefore

$$P_z(Z_n \in N_w \text{ i.o.}) = P(Z_n \in z^{-1} N_w \text{ i.o.}) = P(Z_n \in x^{-1} N_y \text{ i.o.}) = 1.$$

Q.E.D.

Lemma 3.1. *Let* E *be compact. Then for each* N_u *of the unit element* u *in* G *and for each* $f \in F$ *there exists* N_f *of* f *in* F *such that* $N_f e \subset N_u(fe)$ *for all* $e \in E$.

Proof. Let $N \equiv N_u$ and $f \in F$ be given. For each $e \in E$, there exist N_{fe} in G, N_f in F, N_e in E, such that:

$$\begin{aligned} &N_{fe} N_{fe}^{-1} \subset N,\ N_f N_e \subset N_{fe}, \text{ and hence, for every } e' \in N_e, \\ (3) \qquad &fe' \subset N_{fe},\ N_{fe}(fe')^{-1} \subset N,\ N_f N_e \subset N_{fe} \subset N(fe') \text{ for all } e' \in N_e. \end{aligned}$$

Since a finite collection of N_e's cover E, say $N_{e_1}, N_{e_2}, \ldots, N_{e_n}$, then the desired neighborhood is $N_f = \bigcap N_f^{(i)}$, where $N_f^{(i)}$ is the neighborhood of f in F chosen above relative to N_{e_i}, $i = 1, 2, \ldots, n$. Q.E.D.

Proposition 3.2. *If* $x \to y$ *i.o. and* $x \to z$, *then* $z \to y$ *i.o.*

Proof. (i) Suppose $z \not\to y$ i.o. Then $P_z(Z_n \in \tilde{N}_y \text{ f.o.}) = P(Z_n \in z^{-1}\tilde{N}_y \text{ f.o.}) > \varepsilon > 0$ for some neighborhood $\tilde{N}_y$ of y. Choose $H = C \times G \times F$ where $C \subset E$ is compact such that $P(Z_1 \in H^c) < \varepsilon/2$. This implies that $P(Z_n \in H^c \text{ i.o.}) < \varepsilon/2$, since the set $\{Z_n \in H^c \text{ i.o.}\}$ is contained in $\{Z_1 \in H^c\}$.

(ii) Suppose $x = (e, g_x, f_x)$, $y = (e, g_y, f_y)$, $z = (e, g_z, f_z)$, and $\tilde{N}_y = N_e \times \tilde{N}_{g_y} \times \tilde{N}_{f_y}$, $N_y = N_e \times N_{g_y} \times N_{f_y}$, $N_z = N_e \times N_{g_z} \times N_{f_z}$, where N_e, N_{f_y}, $\tilde{N}_{f_y}$, and N_{f_z} are neighborhoods of e, f_y and f_z in E and F respectively, and $N_{g_y} = g_y N_u$, $\tilde{N}_{g_y} = g_y \tilde{N}_u$ and $N_{g_z} = g_z \hat{N}_u$ with N_u, $\tilde{N}_u$, $\hat{N}_u$ being neighborhoods of the identity u of G. It is easy to compute

(4) $$z^{-1}\tilde{N}_y = \{(a,b,c) \in S; c \in \tilde{N}_{f_y} \text{ and } b \in (f_z a)^{-1} g_z^{-1} g_y \tilde{N}_u\},$$

(5) $$N_z^{-1} N_y = \{(a,b,c) \in S; c \in N_{f_y} \text{ and } b \in (N_{f_z} a)^{-1} \hat{N}_u^{-1} g_z^{-1} g_y N_u\}.$$

(iii) Write $(z^{-1}\tilde{N}_y) \cap H = A$, $(N_z^{-1}N_y) \cap H = B$. Since C is compact, by (4) and (5) and Lemma 3.1, for given z and $\tilde{N}_y$, we can find N_{f_y}, N_{f_z}, $\hat{N}_u$ and N_u such that $B \subset A$.

(iv) Since $x \to z$, there exists a $k > 0$ such that $P_x(Z_k \in N_z) = P(xZ_{k-1} \in N_z) > 0$. Then

$$\begin{aligned}
P_x(Z_n \in N_y \text{ f.o.}) &= P(xZ_n \in N_y \text{ f.o.}) \\
&\geq P(xZ_{k-1} \in N_z \text{ and } (xZ_{k-1})^{-1}(xZ_n) \in (xZ_{k-1})^{-1}N_y \text{ f.o.}) \\
&\geq P(xZ_{k-1} \in N_z \text{ and } Z_k^{-1} Z_n \in N_z^{-1} N_y \text{ f.o.}) \\
&\geq P(xZ_{k-1} \in N_z) \cdot P(Z_n \in N_z^{-1} N_y \text{ f.o.}) \\
&\geq P(xZ_{k-1} \in N_z)[P(Z_n \in N_z^{-1} N_y \cap H \text{ f.o.}) - P(Z_n \in H^c \text{ i.o.})] \\
&\geq P(xZ_{k-1} \in N_z)[P(Z_n \in B \text{ f.o.}) - \varepsilon/2] \\
&\geq P(xZ_{k-1} \in N_z)[P(Z_n \in A \text{ f.o.}) - \varepsilon/2] \\
&\geq P(xZ_{k-1} \in N_z)[P(Z_n \in z^{-1}\tilde{N}_y \text{ f.o.}) - \varepsilon/2] > 0
\end{aligned}$$

This is a contradiction. Q.E.D.

Lemma 3.2. *If a subsemigroup D of a completely simple semigroup $S = E \times G \times F$ has a minimal right (or left) ideal (of itself), then D is also completely simple.*

Proof. By hypothesis, for some $a \in D$, aD is right simple and being left cancellative ($aD \subset aS =$ a right group), aD is a right group and contains an idempotent. By [4, II, p. 88 or I, p. 84], D has a completely simple kernel $K = E' \times G' \times F'$, $E' \subset E$, $F' \subset F$, $G' \subset G$, and G' is a group. Let $(e,g,f) \in D$. Then $e \in E'$, $f \in F'$ by the ideal property of K. Hence $fe \in G'$ and so $(fe)^{-1} \in G'$ so that $(e,(fe)^{-1},f) \in K$ and also $(e,g,f) \cdot (e,(fe)^{-1},f) = (e,g,f) \in K$. Hence $K = D$.

Proposition 3.3. *If $x \to y$ i.o., then xD is a closed right group and D is completely simple.*

Proof. By Proposition 3.2, $y \in \bigcap_{z \in \overline{xD}} \overline{zD} \equiv I$. We claim that I is a minimal right ideal of D and hence by Lemma 3.2, D is completely simple. We only need to prove that I is right simple, i.e., $sI = I$ for all $s \in I$. Let $s \in I$ and let $w \in sI \subset I \subset \overline{xD} \subset xS =$ a right group. Then sI is a right ideal of D and

$\overline{sI} = sI$ since the left translations in a right group are closed. Hence $I \subset \overline{wD} \subset sI \subset I$. This completes the proof.

In the representation of $S = E \times G \times F$ we may choose:

$E \equiv \mathcal{E}(Se) \supset E' \equiv \mathcal{E}(De)$, where $e \in \mathcal{E}(D)$;

$F \equiv \mathcal{E}(eS) \supset F' \equiv \mathcal{E}(eD)$;

$G \equiv eSe \supset G' \equiv eDe$, so that

$D = E' \times G' \times F' \subset S = E \times G \times F$.

This representation will be used in the proof of Theorem 3.2 in the sequel.

Definition 3.2. We write $G_{ef} \equiv \{e\} \times G \times \{f\}$ = a typical maximal subgroup of $S \equiv E \times G \times F$, and $u_{ef} \equiv (e,(fe)^{-1},f)$ = the unit of G_{ef}.

Theorem 3.1. *Let $S = E \times G \times F$. Then for each idempotent $u_{ef} \in S$, $u_{ef} \to u_{ef}$ i.o. $\Leftrightarrow \sum_{n=1}^{\infty} P_{u_{ef}}(Z_n \in N) = \infty$ for every neighborhood N of u_{ef}.*

Proof. The "$\Rightarrow$" part is trivial by the Borel-Cantelli lemma. We shall prove the "$\Leftarrow$" part in four steps.(4)

Step I. Let k be a fixed positive integer. Let N be a given neighborhood of u_{ef}. Since

$$\infty = \sum_{n=1}^{\infty} P_{u_{ef}}(Z_n \in N) = \sum_{j=1}^{k} \sum_{i=0}^{\infty} P_{u_{ef}}(Z_{j+ik} \in N),$$

there is a j_0 such that $1 \leq j_0 \leq k$ and $\sum_{i=0}^{\infty} P_{u_{ef}}(Z_{j_0+ik} \in N) = \infty$. Now

$$
\begin{aligned}
1 &\geq P_{u_{ef}}(Z_n \in N \text{ for finitely many } n) \\
&\geq \sum_{i=0}^{\infty} P_{u_{ef}}(Z_{j_0+ik} \in N, Z_n \notin N \text{ for all } n \geq j_0 + (i+1)k) \\
&\qquad\qquad \text{(these sets are pairwise disjoint)} \\
&\geq \sum_{i=0}^{\infty} P_{u_{ef}}(Z_{j_0+ik} \in N, Z_{j_0+ik}^{-1} Z_n \notin N^{-1}N \text{ for all } n \geq j_0 + (i+1)k) \\
&= \sum_{i=0}^{\infty} P_{u_{ef}}(Z_{j_0+ik} \in N) P(Z_n \notin N^{-1}N \text{ for all } n \geq k).
\end{aligned}
\tag{6}
$$

Hence

$$P(Z_n \notin N^{-1}N \text{ for all } n \geq k) = 0. \tag{6'}$$

(This result is true for any arbitrary neighborhood N' in $\{e\} \times G \times F$ of u_{ef}.)

Step II. We find $N_0 \subset N$ such that $N_0 = \{(e,b,c); c \in N_f, b \in N_u(ce)^{-1}\}$, where N_f is a neighborhood of f and N_u is a neighborhood of the identity u in the group G. Then it can be checked easily that

(4) In Step I, we show $P(Z_n \notin N^{-1}N$ for all $n \geq k) = 0$. But transition from $N^{-1}N$ to $u_{ef}^{-1}N$ involves complications and this is demonstrated in Steps II and III. Then Step IV completes the proof.

$$u_{ef}^{-1} N_0 = \{(e', b, c); b \in (fe')^{-1}(fe)N_u(ce)^{-1}, c \in N_f, e' \in E\}.$$

Let E_1 be a compact subset of E.

We can find neighborhoods $N'_f \subset N_f$, $N'_u \subset N_u$ of f and u respectively such that

$$N_1 = \{(e', b, c); c \in N'_f, b \in N'_u(ce')^{-1}, e' \in E_1\} \subset u_{ef}^{-1} N_0$$

(see computation of $u_{ef}^{-1} N_0$ above). This is possible because for every $e' \in E_1$ (which is compact), $(fe)^{-1}(fe')u(fe')^{-1}(fe) \subset N_u$,(5) so that we can find N'_f, N'_u such that for every $e' \in E_1$ and, for every $c \in N'_f$, $(fe)^{-1}(fe')N'_u(ce')^{-1}(ce) \subset N_u$, or $N'_u(ce')^{-1} \subset (fe')^{-1}(fe)N_u(ce)^{-1}$, for every $c \in N'_f$ and every $e' \in E_1$.

Now we can find $N''_f\ (\subset N'_f)$, $N''_u\ (\subset N'_u)$ such that

$$N_2 = \{(e', b, c); c \in N''_f, b \in N''_u(ce')^{-1}, e' \in E_1\} \subset N_1,$$

and

$$N_2^{-1} N_2 \cap E_1 \times G \times F \subset N_1.$$

This is possible since

$$\begin{aligned} &N_2^{-1} N_2 \cap E_1 \times G \times F \\ &\quad = \{(e', b, c); (e'', b', c')(e', b, c) \in N_2, \text{ where } (e'', b', c') \in N_2, e' \in E_1\} \\ &\quad = \{(e', b, c); e' \in E_1, b'(c'e')b \in N''_u(ce'')^{-1}, \\ &\qquad\qquad \text{where } e'' \in E_1, c \in N''_f, c' \in N''_f, b' \in N''_u(c'e'')^{-1}\} \\ &\quad \subset \{(e', b, c); e' \in E_1, b \in (c'e')^{-1}(c'e'')N''^{-1}_u N''_u(ce'')^{-1}, \\ &\qquad\qquad \text{where } c' \in N''_f, e'' \in E_1, c \in N''_f\} \end{aligned}$$

which can be seen to be a subset of N_1 by properly choosing N_f'' and N_u'' since for every e', $e'' \in E_1$, $(fe')^{-1}(fe'')u(fe'')^{-1}(fe') \subset N'_u$, so that there exist N''_f, N''_u such that

$$(N''_f e')^{-1}(N''_f e'')N''^{-1}_u N''_u(N''_f e'')^{-1}(N''_f e') \subset N'_u.$$

Step III. We claim that for all k (a positive integer)

$$P_{u_{ef}}(Z_n \notin N \text{ for all } n \geq k) = 0.$$

To prove this, let $\varepsilon > 0$. Let E_1 be a compact subset of E such that $\mu((E - E_1) \times G \times F) < \varepsilon$. By *Step* II, we can find N_2, a relative neighborhood of u_{ef} in $\{e\} \times G \times F$, such that $N_2^{-1} N_2 \cap E_1 \times G \times F \subset u_{ef}^{-1} N$. Now,

(5) Note that the mapping $(f, e) \to f \cdot e$ from $F \times E \to G$ is continuous by the definition of a completely simple semigroup.

$$\begin{aligned}P_{u_{ef}}(Z_n \notin N \text{ all } n \geq k) &\leq P(Z_n \notin u_{ef}^{-1} N \text{ all } n \geq k)\\ &\leq P(Z_n \notin N_2^{-1} N_2 \cap E_1 \times G \times F \text{ all } n \geq k)\\ &\leq P(Z_n \notin N_2^{-1} N_2 \text{ all } n \geq k)\\ &\quad + P(Z_n \in (E - E_1) \times G \times F \text{ for some } n)\\ &= 0 + \varepsilon \quad (\text{by } \textit{Step } \text{I}).\end{aligned}$$

(Note that $\{Z_n \in (E - E_1) \times G \times F$ for some $n\}$ is contained in $\{Z_1 \in (E - E_1) \times G \times F\}$.)

Since $\varepsilon > 0$ is arbitrary, our claim is proven.

Step IV. We have

$$\begin{aligned}&P_{u_{ef}}(Z_n \in N \text{ for finitely many } n)\\ &\quad = \sum_{k=1}^{\infty} P_{u_{ef}}(Z_k \in N, Z_n \notin N \text{ for all } n \geq k+1)\\ &\quad \leq \sum_{k=1}^{\infty} P_{u_{ef}}(Z_n \notin N \text{ for all } n \geq k+1) = 0.\end{aligned}$$

Hence $P_{u_{ef}}(Z_n \in N \text{ i.o.}) = 1$. Q.E.D.

Lemma 3.3. *In any semigroup whenever $R \neq \varnothing$, R is a left ideal of D.*

It follows from the fact that $P_x(Z_n \in N_x \text{ i.o.}) = 1 \Rightarrow P(xZ_n \in N_x \text{ i.o.}) = 1$ which implies $P(yxZ_n \in yN_x \text{ i.o.}) = 1$ and every neighborhood $N_{yx} \supset yN_x$ for some N_x.

Lemma 3.4. *If $x \to y$ i.o. for some x, y then $xD \subset R$.*

Proof. By Proposition 3.3, xD is a closed right group. Let $\mathcal{E}(xD)$ be the set of idempotents of xD and let $f \in \mathcal{E}(xD)$. Since $y \to f$, for each neighborhood N of f there is k such that $P_y(Z_k \in N) > 0$; by lower semicontinuity in x of the function $\mu^k(x^{-1}N)$ (e.g., see [7, p. 143]), there is a neighborhood of y, N_y, such that $P_{x'}(Z_k \in N) > \delta > 0$, for all $x' \in N_y$. Now for $n > k$,

$$P_f(Z_n \in N) = \int P_f(Z_{n-k} \in dx') P_{x'}(Z_{n-k}^{-1} Z_n \in N) \geq \delta P_f(Z_{n-k} \in N_y).$$

By Proposition 3.2, $f \to y$ i.o., so $\sum P_f(Z_n \in N_y) = \infty$ by the Borel-Cantelli lemma. Hence by Theorem 3.1 and the above inequality, $f \in R$ for every $f \in \mathcal{E}(xD)$. Since xD is a right group, $xD = \bigcup\{xDf;$ where $f \in \mathcal{E}(xD)\}$. Since R is a left ideal of D (Lemma 3.3), we have $xD \subset R$. Q.E.D.

Theorem 3.2. (i) *If $x \to y$ i.o. for some x, y, then $R = D =$ a completely simple semigroup and for every pair $z, w \in aD$, $a =$ any element of D, we have $w \to z$ i.o.*

(ii) *Either $R = \varnothing$ or $R = D =$ a completely simple subsemigroup.*

Proof. By Proposition 3.3, $D = E \times G \times F$ is completely simple and by Lemma 3.4, there is an idempotent $u_{ef} = (e,(fe)^{-1},f) \in R$. Since (e',g,f) $(e,(fe)^{-1},f) = (e',g,f) \in R$ (since R is a left ideal of D by Lemma 3.3), we obtain $G_{e'f} \equiv \{e'\} \times G \times \{f\} \subset R$ for all $e' \in E$. Since $(e,(fe)^{-1},f)(e,g,f')$ $= (e,g,f') \subset u_{ef}D \subset R$ by Lemma 3.4, $G_{ef'} \subset R$ for every $f' \in F$. Consider next $G_{e'f'}$, for any $e' \in E, f' \in F$. Since the idempotent $(e',(fe')^{-1},f) \in R$, again by the above argument, $G_{e'f'} \subset R$ and hence $R = D$. The rest of the claims follow easily from Proposition 3.2.[6] Q.E.D.
the left random walk.

Theorem 3.3. *$R \neq \varnothing \Leftrightarrow \sum_{n=1}^{\infty} P(Z_n \in N) = \infty$ for all open neighborhoods N of some $x \in D$.*

Proof. If $R \neq \varnothing$, then $R = D$ by Theorem 3.2. Hence $\sum P_y(Z_n \in N) = \infty$ for all $y \in D \cap N$. Choose k such that $\mu^k(D \cap N) > 0$. Then

$$\sum_{n=1}^{\infty} P(Z_n \in N) \geq \sum_{n=k+1}^{\infty} \int_D P_y(Z_{n-k} \in N)\mu^k(dy)$$
$$= \int \sum_{n=k+1}^{\infty} P_y(Z_{n-k} \in N)\mu^k(dy) = \infty.$$

We prove next the "$\Leftarrow$" part: Let $\sum_{n=1}^{\infty} P(Z_n \in N) = \infty$ for all neighborhoods N of some $x' \in D$. Then for each neighborhood N of $x' = (e,g,f)$, we have as in the proof of Theorem 3.1, for every positive integer k,

$$P(Z_n \notin N^{-1}N \text{ for all } n \geq k) = 0.$$

Now given an open neighborhood $N_1 = N_0(e) \times N_0((fe)^{-1}) \times N_0(f)$ of $(e,(fe)^{-1}, f)$, we can find an open neighborhood $N = N_0(e) \times g(fe)N_0((fe)^{-1}) \times N_0(f)$ of $x' = (e,g,f)$ such that $N^{-1}N = N_1^{-1}N_1$. The reason is as follows: Let $(x,y,z) \in N^{-1}N$. Then there exist $e' \in N_0(e)$, $g' \in N_0((fe)^{-1})$, $f' \in N_0(f)$ such that

$$(e',g(fe)g',f')(x,y,z) \in N \quad \text{or} \quad (e',g(fe)g'(f'x)y,z) \in N.$$

This means that $g'(f'x)y \in N_0((fe)^{-1})$ or $(e',g',f')(x,y,z) \in N_1$ or $(x,y,z) \in N_1^{-1}N_1$. Hence $N^{-1}N \subset N_1^{-1}N_1$. Similarly $N_1^{-1}N_1 \subset N^{-1}N$. Hence for every positive integer k, $P(Z_n \notin N_1^{-1}N_1$ for all $n \geq k) = 0$, for every open neighborhood N_1 of $(e,(fe)^{-1},f)$. Now by following the proofs of Steps II, III, and IV in Theorem 3.1, we can prove that for every neighborhood N_1 of $(e,(fe)^{-1},f) \equiv u_{ef}$,

$$P_{u_{ef}}(Z_n \in N_1 \text{ i.o.}) = 1.$$

Hence $u_{ef} \in R$ and $R \neq \varnothing$. Q.E.D.

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(6) If in particular S = a right group and $R \neq \varnothing$, then $R' = R = R_l = D$ = a right subgroup of S but R_l' may be empty (cf. Example 2.1), where R_l', R_l are the corresponding recurrence sets for

Corollary 3.1. *$R \neq \emptyset \Leftrightarrow \sum P_x(Z_n \in N) = \infty$ for all neighborhoods N of some $y \in D$, x being some fixed element of D.*

For proof, observe that $\sum P_x(Z_n \in N_y) = \infty$ for all N_y *implies* that $y \in \text{Closure}\,(xD) \subset \{e\} \times G \times F$, where $x = (e,g,f)$. Also equations (6) and (6′) in Theorem 3.1 are valid with $P_{u_{ef}}$ replaced by P_x-measure and the last part of the proof of Theorem 3.3 applies in this case.

Definition 3.3. We say that Z_n is a *recurrent* right *random walk* if $R \neq \emptyset$.

Theorem 3.4. *Suppose that the group factor G is compact in the representation of $S = E \times G \times F$. Then $\{Z_n\}$ is always recurrent and (as in Theorem 3.2) $R = D =$ a completely simple subsemigroup.*

Proof. Let J and H be compact subsets of E and F such that $\mu(J \times G \times H) > 0$. (This can be done by the regularity of μ.) Let $L = E \times G \times H$ and $L_1 = J \times G \times H$; then for any $x \in L_1 \cap D$,

$$\begin{aligned} P_x(Z_n \in L_1 \text{ i.o.}) &= \lim_{k\to\infty} P_x\left(\cup_{n=k}^{\infty}\{Z_n \in L\}\right) \\ &= \lim_{k\to\infty} P_x\left(\cup_{n=k}^{\infty}\{X_n \in L\}\right) \\ &= \lim_{k\to\infty}\left[1 - P_x\left(\cap_{n=k}^{\infty}\{X_n \notin L\}\right)\right] \\ &= \lim_{k\to\infty}\left[1 - \prod_{n=k}^{\infty} P_x\{X_n \notin L\}\right] = 1, \end{aligned}$$

because $P(X_n \notin L) = \mu(S - L) < 1$, for all n. Hence for each subset $J \subset E$ such that $\mu(J \times G \times H) > 0$, we have

$$\begin{aligned} P(Z_n \in J \times G \times H \text{ i.o.}) &= \int_{(E\times G\times F)\cap D} P_x(Z_n \in J \times G \times H \text{ i.o.})\,\mu(dx) \\ &\geq \int_{(J\times G\times H)\cap D} P_x(Z_n \in J \times G \times H \text{ i.o.})\,\mu(dx) \\ &= \mu(J \times G \times H) > 0. \end{aligned}$$

Now suppose that Z_n is not recurrent. For every $x \in D \cap J \times G \times H$, there exists a neighborhood $N(x)$ such that $\sum P(Z_n \in N(x)) < \infty$, by Theorem 3.3. By the Borel-Cantelli lemma, $P(Z_n \in N(x) \text{ i.o.}) = 0$. Since $J \times G \times H$ is compact, we can find $x_1, x_2, \ldots x_m \in J \times G \times H$ and neighborhoods $N(x_1)$, $N(x_2) \cdots$, $N(x_m)$ such that

$$J \times G \times H \subset \cup_{i=1}^{m} N(x_i)$$

and

$$P(Z_n \in J \times G \times H \text{ i.o.}) \leq \sum_{k=1}^{m} P(Z_n \in N(x_k) \text{ i.o.}) = 0$$

which is a contradiction. Q.E.D.

Theorem 3.5. *Let $S = E \times G \times F$. Then S can support a recurrent random walk (i.e., a walk $\{Z_n\}$ with $R \neq \emptyset$) $\Leftrightarrow$ G can support a recurrent random walk.*

Proof. The "$\Leftarrow$" part: It is trivial; just restrict D in $\{e\} \times G \times \{f\}$, and use Definition 3.1 and Theorem 3.2. The "$\Rightarrow$" part: Let $x = (e, g_x, f_x) \in D$ and consider $\{e\} \times G \times F$, Define $U_g \equiv \{(e, g(ce)^{-1}, c); c \in F\}$. Then

$$U_{g_1} U_{g_2} = \{(e, g_1(ce)^{-1}, c)(e, g_2(c'e)^{-1}, c'); c, c' \in F\}$$
$$= \{(e, g_1 g_2(c'e)^{-1}, c'); c' \in F\} = U_{g_1 g_2}.$$

Let $\mathcal{U} = \{U_g; g \in G\}$. It can be shown that $\mathcal{U}$ is a group isomorphic to G. Recall that $Z_n = X_1 X_2 \cdots X_n$. We define

$$V_n = U_g \qquad \text{iff } X_n \in U_g \text{ for } n = 1, 2, \ldots,$$
$$W_n = V_1 V_2 \cdots V_n, \quad \text{for } n = 1, 2, \ldots.$$

We note (from the definition) that the V_i's are independent because the X_i's are independent. By the group property of the U_g's,

$$W_n = U_g \quad \text{if and only if} \quad Z_n = X_1 X_2 \cdots X_n \in U_g.$$

Hence $\{W_n\}$ is a random walk on $\mathcal{U} \cong G$. ($W_n = V_1 V_2 \cdots V_n$ and the V_i's are independent.) We note that

(i) $P(W_n \in E_A) = P(Z_n \in \cup E_A)$, where A is a Borel subset of G and $E_A = \{U_g; g \in A\}$, $\cup E_A = \cup_{g \in A} U_g$ and $P(W_n \in E_A \text{ i.o.}) = P(Z_n \in \cup E_A \text{ i.o.})$.

(ii) If $\{Z_n\}$ starts at $x = (e, g_x, f_x)$, then $\{W_n\}$ starts at U_{g_x}.

By (i) and (ii), $\{W_n\}$ is a recurrent right random walk on $\mathcal{U} \cong G$. Q.E.D.

(*Note.* In the definition of W_n above we may define $V_n = g$ if $X_n \in U_g$ since $\mathcal{U}$ and G are homeomorphic (given identical topologies); then if A is open in G, U_A is open in $\{e\} \times G \times F$ and since Z_n starts at x, every open set may be restricted in $\{e\} \times G \times F$ and $P_x(Z_n \in U_A \text{ i.o.}) = 1$ since $x \in R$.)

4. Example 2.2 suggests that when S is a compact semigroup the following result holds.

Lemma 4.1. *Let S be a compact semigroup and D the support of the random walks induced by a measure μ. Let K be the kernel of D. Then for any neighborhood U of K and any $x \in D$*

$$P(Z_n \in U \text{ i.o.}) = P_x(Z_n \in U \text{ i.o.}) = 1.$$

Proof. There exists an open ideal I such that $K \subset I \subset U$ by a result of R. J. Koch and A. D. Wallace [5]. There is k such that $\mu^k(I) > 0$. We observe that

$$(7) \qquad \mu^{n+k}(I^c) = \int_{I^c} \mu^n(I^c x^{-1})\mu^k(dx) \le \mu^n(I^c)\mu^k(I^c); \qquad I^c = D - I.$$

Also

$$\sum \mu^n(I^c) = (\text{Constant}) + \mu^k(I^c) + \cdots + \mu^{2k-1}(I^c) + \mu^{2k}(I^c) + \cdots + \mu^{3k-1}(I^c) + \cdots$$
$$\le (\text{Constant}) + k\mu^k(I^c) + k[\mu^k(I^c)]^2 + k[\mu^k(I^c)]^3 + \cdots < \infty.$$

Hence by the Borel-Cantelli lemma, $P(Z_n \in I^c \text{ i.o.}) = 0$ and $P(Z_n \in I \text{ i.o.}) = 1$. (Observe that also $\sum \mu^n(I^c x^{-1}) \le \sum \mu^n(I^c) < \infty$ for every $x \in D$.) Q.E.D.

Theorem 4.1. *Let S be a compact abelian semigroup and let K be the kernel of D, the support of the random walk $\{Z_n\}$ on S. Let R', R be the sets of states which are recurrent in the sense of (1) and Definition 3.1 respectively. Then $R' = R = K$.*

Proof. Since S is abelian, K is a compact topological group. For any (relatively open) neighborhood $N \subset D$, by restricting our attention to D replacing S by D,

$$(8) \qquad P(Z_n \in N \text{ i.o.}) = 0 \text{ or } 1,$$

since $P(z_n \in D^c \text{ for some } n) = 0$. (The proof of (8) (known as Hewitt-Savage 0-1 law) can be carried over to locally compact abelian semigroups from [2, p.236]). Since D is compact, it satisfies

(9) $x \notin yD \Rightarrow$ there exist neighborhoods N_x, N_y such that $N_y^{-1} N_x \cap D = \varnothing$, i.e., $N_y D \cap N_x = \varnothing$.(7)

To prove the theorem we first prove:

(10) $a \in R' \Rightarrow y \to a$ $(a \in yD)$ for every $y \in D$.

(11) $a \in R \Rightarrow y \to a$ (i.e., $a \in yD$) for every $y \in D$.

(12) $a \in R'$ and $a \to b \Rightarrow b \in R'$.

(13) $x \to a$ i.o. and $a \to b \Rightarrow x \to b$ i.o.

Suppose $a \notin bD$ for some $b \in D$; then by (9) $N_b^{-1} N_a = \varnothing$ for some N_a and some N_b, so that $N_b^{-1} N_a^c$ is all of D. Choosing k such that $P(S_k \in N_b) > 0$, a contradiction to the recurrence property of a in (10) and (11) will obtain from the following relations respectively:

$$P(Z_n \in N_a^c \text{ eventually}) \ge P(Z_n \in N_a^c \text{ eventually}, Z_k \in N_b)$$
$$\ge P(Z_k^{-1} Z_n \in N_b^{-1} N_a^c \text{ eventually}, Z_k \in N_b) > 0$$

(7) For, if $N_y D \cap N_x \ne \varnothing$ for all N_x, N_y, then we can find nets $y_\alpha \in N_y$, $x_\alpha \in N_x$ and $d_\alpha \in D$ such that $y_\alpha d_\alpha = x_\alpha$ and $y_\alpha \to y$, $x_\alpha \to x$ which implies $x \in yD$.

and

$$P(xZ_k X_{k+1} \cdots X_{k+n} \in N_a^c \text{ eventually})$$
$$= P(Z_k x X_{k+1} \cdots X_{k+n} \in N_a^c \text{ eventually})$$
$$\text{(by the abelian property of } S\text{)}$$
$$\geq P(Z_k x X_{k+1} \cdots X_{k+n} \in N_a^c \text{ eventually}, Z_k \in N_b)$$
$$\geq P(x X_{k+1} \cdots X_{k+n} \in N_b^{-1} N_a^c \text{ eventually}, Z_k \in N_b)$$
$$> 0.$$

To prove (12) and (13), suppose that $a \to b$, i.e., $b = ad$ for some $d \in D$. Choose N_a, N_d such that $N_a N_d \subset N_b$ and choose k such that $P(Z_k \in N_d) > 0$. Then (12) and (13) follow from the following relations in conjunction with (8) and stationarity:

$$P(Z_n \in N_b \text{ i.o.}) \geq P(Z_n \in N_a N_d \text{ i.o.})$$
$$\geq P(Z_k \in N_d) P(Z_k^{-1} Z_n \in N_a \text{ i.o.}) > 0$$

and

$$P(Z_k x X_{k+1} \cdots X_{k+n} \in N_b \text{ i.o.}) \geq P(Z_k \in N_d) P(x X_{k+1} \cdots X_{k+n} \in N_a \text{ i.o.}) > 0.$$

(Note that $P(X_{k+1} \cdots X_{k+n} \in x^{-1} N_a \text{ i.o.}) = P(Z_n \in x^{-1} N_a \text{ i.o.}) = P(xZ_n \in N_a \text{ i.o.}) = P_x(Z_n \in N_a \text{ i.o.})$.)

Next we show that $R' \neq \varnothing$ implies $R \neq \varnothing$. Suppose $a \in R' = K$. By integration, $P_x(Z_n \in N_a \text{ i.o.}) = 1$ for some $x \in D$. Hence for $g \in K$, $P_x(Z_n \in N_a \text{ i.o.}) = P(xZ_n \in N_a \text{ i.o.}) = 1 = P(gxZ_n \in gN_a \text{ i.o.}) = P_{gx}(Z_n \in gN_a \text{ i.o.}) = 1$. Hence we may assume that $x \to y$ i.o. for some $x, y \in K$. Hence by using (13), $R \neq \varnothing$.

Next assume $R' = \varnothing$. Then for each $x \in K$, there is a neighborhood N_x of x such that $P(Z_n \in N_x \text{ i.o.}) < 1$ and hence $P(Z_n \in N_x \text{ i.o.}) = 0$ by (8). Since K is compact, there is a finite cover of K consisting of these N_x's. Say $K \subset U = \bigcup_{i=1}^n N_x^{(i)}$. Then $P(Z_n \in U \text{ i.o.}) = 0$, which contradicts Lemma 4.1. Q.E.D.

A remark on invariant measures for Z_n when S is completely simple. Suppose $R \neq \varnothing$. Consider any recurrence class of D, say aD, $a \in D$. We say that the σ-finite measure ν is *invariant* for $\{Z_n\}$ on aD if it is invariant for the restricted transition function $P(x, A) = \mu(x^{-1}A)$ on aD, i.e., $\int_{aD} P(x, \cdot)\nu(dx) = \nu(\cdot)$. By Theorem 3.2,

(14) For any open $N \subset aD$, $P_x(Z_n \in N \text{ i.o.}) = 1$ for every $x \in aD$. But (14) is a "Harris recurrence condition" for *open sets*. By a result of Foguel, condition (14) implies that there is a σ-finite invariant measure for Z_n on aD which is finite on compact sets and positive on open sets. (Actually (14) implies the recurrence

condition used by Foguel and Horowitz in (1) S. R. Foguel, *Existence of a σ-finite invariant measure for a Markov process on a locally compact space*, Israel J. Math. **6** (1968), 1–4; and (2) S. Horowitz, *Markov processes on a locally compact space*, Israel J. Math. **7** (1969), 311–324; see particularly pp. 312, 317, and 318.)

We finally remark that the referee has pointed out that there are two recent announcements by Jean Larisse in C.R. Acad. Sci. Paris Sér. A **274** (1972), pp. 339–341 and 431–416 which are related to our work, but only deal with discrete semigroups.

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